

Therefore, stability in this case is determined by the condition $\sigma < 2$.

The temperature in the friction zone is determined by using (1), here not the asymptotic but the exact value of $\Sigma(\tau)$ corresponding to all poles of $M(\omega)$ should be used.

We present the results of the calculations for $\tau \rightarrow \infty$ in Table 1.

BIBLIOGRAPHY

1. Andreevskii, V. M., Measurement of friction during vibrations. *Izv. VUZ, Fizika*, No.6 (73), 1968.
2. Grigorova, S.R. and Tolstoi, D.M., On resonance drop of friction. *Dokl. Akad. Nauk SSSR*, Vol. 167, No.3, 1966.
3. Kragel'skii, I. V., Friction and Wear, 2nd ed., Moscow, Mashinostroenie, 1968.
4. Korovchinskii, M. V., Local thermal contact with quasistationary heat liberation in the friction process. Sb. "Theory of Friction and Wear", Moscow, "Nauka", 1965.
5. Korovchinskii, M. V., Principles of the theory of thermal contact with local friction. Sb. "New Material in the Theory of Friction", Moscow, "Nauka" 1966.
6. Chichinadze, A. V., Analysis and Investigation of External Friction with Deceleration, Moscow, "Nauka", 1967.
7. Whittaker, E. T. and Watson, G. N., Modern Analysis, Vol.2, 2nd ed., Moscow, Fizmatgiz, 1963.
8. Babich, V. M., Kapilevich, M. B., Mikhlin, S. G., Natanson, G. N., Riz, P. M., Slobodetskii, L. N. and Smirnov, M. M. Linear Equations of Mathematical Physics. Moscow, "Nauka", 1964.
9. Titchmarsh, E., Introduction to the Theory of Fourier Integrals. Moscow-Leningrad, Gostekhizdat, 1948.
10. Slonovskii, N. V., Application of a method of constructing inequalities to Bessel functions. *Izv. VUZ, Matematika*, No.4, 1967.

Translated by M. D. F.

ON THE STEADY MOTIONS OF A GYROSTAT SATELLITE

PMM Vol. 33, No. 1, 1969, pp. 127-131

S. Ia. STEPANOV

(Moscow)

(Received February 6, 1968)

Two families of steady motions of a gyrostatt satellite in a central Newtonian force field are considered. The plane of the (circular) orbit of the center of mass of the satellite is biased relative to the attracting center. Sufficient conditions for stability are derived.

These motions complement the numerous already familiar [1] steady motions of a gyrostatt satellite with the center of the circular orbit coincident with the attracting center. As in the case of the latter motions, the stability conditions in our case differ from those obtained under the restricted formulation of the problem [1] by quantities on the order of β/R^3 relative to the principal terms (l is the characteristic dimension of the satellite, R is the distance from the attracting center). The orbital plane bias is of the order of β/R . These quantities are very small indeed when one is dealing with real artificial earth satellites.

The present study is carried out by the Routh method with the aid of some results obtained by Rumiansev [1].

1. We assume that the coordinate system $O\xi_1\xi_2\xi_3$ with its origin at the attracting center is inertial. To the satellite we attach the coordinate system $Gx_1x_2x_3$, directing its axes along the principal axes of inertia. We also introduce the orbital coordinate system $Gy_1y_2y_3$, whose axis y_3 is directed along OG and whose axis y_1 is parallel to the plane $O\xi_2\xi_3$ and points in the direction of motion. All of the coordinate systems are right-handed and rectangular.

The position of the satellite body in the coordinate system $O\xi_1\xi_2\xi_3$ will be defined in terms of the spherical coordinates R, α, σ of the center of mass G of the satellite,

$$\xi_1 = R \cos \alpha \sin \sigma, \quad \xi_2 = R \sin \alpha, \quad \xi_3 = R \cos \alpha \cos \sigma$$

and in terms of the Euler angles θ, ψ, φ , defining the position of the coordinate system $Gx_1x_2x_3$ relative to $Gy_1y_2y_3$.

The projections of the gyrostatic moment k_1, k_2, k_3 on the axes x_1, x_2, x_3 are assumed constant.

2. The problem of finding the steady motions of a gyrost satellite which constitute the relative equilibria of the satellite in the orbital coordinate system and that of determining the conditions of stability of these motions reduce to the determination of the stationary points and conditions of minimal altered potential energy [1]

$$W(R, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2) = \frac{1}{2} K^2 / S - U$$

$$K = k - k_1\beta_1 - k_2\beta_2 - k_3\beta_3, \quad S = MR^2 \cos^2 \alpha + A_1\beta_1^2 + A_2\beta_2^2 + A_3\beta_3^2$$

$$U = \mu M / R - \frac{1}{2} \mu R^{-3} [A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2 - \frac{1}{2}(A_1 + A_2 + A_3)]$$

$$\beta_3 = \sqrt{1 - \beta_1^2 - \beta_2^2}, \quad \gamma_3 = \sqrt{1 - \gamma_1^2 - \gamma_2^2}$$

Here U is the force function; M, A_1, A_2, A_3 are the mass and the central moments of inertia of the satellite; μ is the gravitational constant; k is the constant of the area integral corresponding to the cyclical coordinate σ ; $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the axes ξ_3 and y_3 in the coordinate system $Gx_1x_2x_3$. The variables $\beta_1, \beta_2, \gamma_1, \gamma_2, \alpha$ are related by the expression

$$\chi = \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 - \sin \alpha = 0 \quad (2.1)$$

Introducing the function $W = W + \lambda\chi$ (λ is a Lagrange multiplier), we can rewrite the equations (in addition to (2.1)) of steady motion of a gyrost satellite in terms of $R, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \lambda$ in the form

$$\frac{\partial W_1}{\partial R} = -\frac{K^2}{S^2} MR \cos^2 \alpha + M \frac{\mu}{R^3} - \frac{9}{2} \frac{\mu}{R^4} \left[(A_1 - A_2) \gamma_1^2 + (A_2 - A_3) \gamma_2^2 + \frac{2A_3 - A_1 - A_2}{3} \right] = 0$$

$$\frac{\partial W_1}{\partial \alpha} = \frac{K^2}{S^2} MR^2 \sin \alpha \cos \alpha - \lambda \cos \alpha = 0$$

$$\frac{\partial W_1}{\partial \beta_1} = \frac{K}{S} \left(-k_1 + k_3 \frac{\beta_1}{\beta_3} \right) - \frac{K^2}{S^2} (A_1 - A_2) \beta_1 + \lambda \left(\gamma_1 - \gamma_3 \frac{\beta_1}{\beta_3} \right) = 0$$

$$\frac{\partial W_1}{\partial \beta_2} = \frac{K}{S} \left(-k_2 + k_3 \frac{\beta_2}{\beta_3} \right) - \frac{K^2}{S^2} (A_2 - A_3) \beta_2 + \lambda \left(\gamma_2 - \gamma_3 \frac{\beta_2}{\beta_3} \right) = 0 \quad (2.2)$$

$$\frac{\partial W_1}{\partial \gamma_1} = 3 \frac{\mu}{R^3} (A_1 - A_2) \gamma_1 + \lambda \left(\beta_1 - \beta_3 \frac{\gamma_1}{\gamma_3} \right) = 0$$

$$\frac{\partial W_1}{\partial \gamma_2} = 3 \frac{\mu}{R^3} (A_2 - A_3) \gamma_2 + \lambda \left(\beta_2 - \beta_3 \frac{\gamma_2}{\gamma_3} \right) = 0$$

In addition to those already considered [1] for $\kappa = 0$, Eqs. (2.1) (2.2) also have solutions for $\kappa \neq 0$,

$$R = R_0, \kappa = \kappa_0, \beta_1 = 0, (\beta_2 = \cos(\theta_0 + \kappa_0)), \beta_3 = \sin(\theta_0 + \kappa_0) \quad (2.3)$$

$$\lambda = MR_0^2 \omega_0^2 \sin \kappa_0, \gamma_1 = 0, \gamma_2 = -\sin \theta_0, (\gamma_3 = \cos \theta_0)$$

if the constants $R_0, \kappa_0, \theta_0, \omega_0$ are related by the expressions

$$\omega_0 [k_2 \sin(\theta_0 + \kappa_0) - k_3 \cos(\theta_0 + \kappa_0)] + 1/2 \omega_0^2 (A_2 - A_3) \sin 2(\theta_0 + \kappa_0) + 3/2 \mu R_0^{-2} (A_2 - A_3) \sin 2\theta_0 = 0$$

$$\omega_0^2 \cos^2 \kappa_0 = \frac{\mu}{R_0^2} \left\{ 1 - \frac{9}{2MR_0^2} \left[(A_2 - A_3) \sin^2 \theta_0 + \frac{2A_2 - A_1 - A_3}{3} \right] \right\} \quad (2.4)$$

$$\sin 2\kappa_0 = \frac{3(A_2 - A_3)}{MR_0^2 \omega_0^2} \frac{\mu}{R_0^2} \sin 2\theta_0, \quad k_1 = 0 \quad \left(\omega_0 = \omega_0' = \frac{K_0}{S_0} \right)$$

Solution (2.3) describes the relative equilibrium of a satellite in an orbital coordinate system rotating at the constant angular velocity ω_0 about the axis ζ_3 ; the straight line OG forms a constant angle κ_0 with the plane $O\zeta_2\zeta_1$. One of the principal axes of inertia x_1 of the satellite is directed along the velocity of motion of the center of mass (the axis y_1), while the two other axes x_2 and x_3 lie in the plane Gy_2y_3 and form the angles θ_0 with the axes y_2 and y_3 , respectively. The angle κ_0 is of the order of μ^2 / R^3 and is maximum for $\theta_0 = 1/2\pi$.

The second partial derivatives of the function W_1 for the values (2.3) are as follows (the missing derivatives are equal to zero):

$$\frac{\partial^2 W_1}{\partial R^2} = \frac{4MR_0^2 \cos^2 \kappa_0 - S_0}{S_0} M \omega_0^2 \cos^2 \kappa_0 - 2M \frac{\mu}{R_0^2} + 18 \frac{\mu}{R_0^2} \left[(A_2 - A_3) \sin^2 \theta_0 + \frac{2A_2 - A_1 - A_3}{3} \right]$$

$$\frac{\partial^2 W_1}{\partial \kappa^2} = \frac{MR_0^2}{S_0} \omega_0^2 (S_0 \cos^2 \kappa_0 + MR_0^2 \sin^2 2\kappa_0)$$

$$\frac{\partial^2 W_1}{\partial \beta_1^2} = MR_0^2 \omega_0^2 \sin \kappa_0 \frac{\sin \theta_0}{\cos(\theta_0 + \kappa_0)} + (A_2 - A_1) \omega_0^2 + \frac{\omega_0 k_2}{\cos(\theta_0 + \kappa_0)}$$

$$\frac{\partial^2 W_1}{\partial \beta_2^2} = MR_0^2 \omega_0^2 \sin \kappa_0 \frac{\sin \theta_0}{\cos^2(\theta_0 + \kappa_0)} + (A_2 - A_3) \omega_0^2 + \frac{\omega_0 k_2}{\cos^2(\theta_0 + \kappa_0)} + \frac{\omega_0^2}{S_0} \left[\frac{MR_0^2 \sin 2\kappa_0}{2 \cos(\theta_0 + \kappa_0)} - (A_2 - A_3) \sin(\theta_0 + \kappa_0) \right]^2$$

$$\frac{\partial^2 W_1}{\partial \gamma_1^2} = -MR_0^2 \omega_0^2 \sin \kappa_0 \frac{\sin(\theta_0 + \kappa_0)}{\cos \theta_0} + 3 \frac{\mu}{R_0^2} (A_1 - A_2)$$

$$\frac{\partial^2 W_1}{\partial \gamma_2^2} = -MR_0^2 \omega_0^2 \sin \kappa_0 \frac{\sin(\theta_0 + \kappa_0)}{\cos^2 \theta_0} + 3 \frac{\mu}{R_0^2} (A_2 - A_3)$$

$$\frac{\partial^2 W_1}{\partial R \partial \kappa} = -\frac{2MR_0^2 \cos^2 \kappa_0 - S_0}{S_0} MR_0 \omega_0^2 \sin 2\kappa_0$$

$$\frac{\partial^2 W_1}{\partial R \partial \beta_2} = \frac{2MR_0 \omega_0^2 \cos^2 \kappa_0}{S_0} \left[\frac{MR_0^2 \sin 2\kappa_0}{2 \cos(\theta_0 + \kappa_0)} - (A_2 - A_3) \sin(\theta_0 + \kappa_0) \right]$$

$$\frac{\partial^2 W_1}{\partial \kappa \partial \beta_2} = -\frac{MR_0^2 \omega_0^2 \sin 2\kappa_0}{S_0} \left[\frac{MR_0^2 \sin 2\kappa_0}{2 \cos(\theta_0 + \kappa_0)} - (A_2 - A_3) \sin(\theta_0 + \kappa_0) \right]$$

$$\frac{\partial^2 W_1}{\partial R \partial \gamma_2} = \frac{9\mu (A_2 - A_3) \sin \theta_0}{R_0^2}, \quad \frac{\partial^2 W_1}{\partial \beta_1 \partial \gamma_1} = MR_0^2 \omega_0^2 \sin \kappa_0$$

$$\frac{\partial^2 W_1}{\partial \beta_2 \partial \gamma_2} = -\frac{MR_0^2 \omega_0^2 \sin^2 \kappa_0}{\cos \theta_0 \cos(\theta_0 + \kappa_0)}$$

Provided the area integral is unperturbed, the sufficient condition [1,2] of stability of steady motion (2.3) with respect to $R, R', \kappa, \kappa', \theta, \theta', \psi, \psi', \varphi, \varphi', \sigma'$ is fulfillment of the Sylvester conditions of positive definiteness of the second variation $\delta^2 W_1$ on substitution into it of the parameter values given by (2.3), i.e. fulfillment of the equation

$$\delta \gamma_3 = \cos \theta_0 \delta \kappa - \frac{\cos \theta_0}{\cos(\theta_0 + \kappa_0)} \delta \beta_3$$

For real artificial satellites ($l \ll R$) these conditions reduce to the three following inequalities:

$$\begin{aligned} s_1 &= \frac{\partial^2 W_1}{\partial \gamma_1^2} > 0, & s_2 &= \frac{\partial^2 W_1}{\partial \gamma_1^2} \frac{\partial^2 W_1}{\partial \beta_1^2} - \left(\frac{\partial^2 W_1}{\partial \gamma_1 \partial \beta_1} \right)^2 > 0 & (2.5) \\ s_3 &= \det | a_{ij} | > 0 & (a_{ij} &= a_{ji}; i, j = 1, 2, 3) \\ a_{11} &= \frac{\partial^2 W_1}{\partial R^2}, & a_{22} &= \frac{\partial^2 W_1}{\partial \kappa^2} + \cos^2 \theta_0 \frac{\partial^2 W_1}{\partial \gamma_3^2} \\ a_{33} &= \frac{\partial^2 W_1}{\partial \beta_3^2} + \frac{\cos^2 \theta_0}{\cos^2(\theta_0 + \kappa_0)} \frac{\partial^2 W_1}{\partial \gamma_3^2} - 2 \frac{\cos \theta_0}{\cos(\theta_0 + \kappa_0)} \frac{\partial^2 W_1}{\partial \beta_3 \partial \gamma_3} \\ a_{13} &= \frac{\partial^2 W_1}{\partial R \partial \kappa} + \cos \theta_0 \frac{\partial^2 W_1}{\partial R \partial \gamma_3}, & a_{12} &= \frac{\partial^2 W_1}{\partial R \partial \beta_3} - \frac{\cos \theta_0}{\cos(\theta_0 + \kappa_0)} \frac{\partial^2 W_1}{\partial R \partial \gamma_3} \\ a_{23} &= \frac{\partial^2 W_1}{\partial \kappa \partial \beta_3} - \frac{\cos^2 \theta_0}{\cos(\theta_0 + \kappa_0)} \frac{\partial^2 W_1}{\partial \gamma_3^2} + \cos \theta_0 \frac{\partial^2 W_1}{\partial \beta_3 \partial \gamma_3} \end{aligned}$$

To within terms of the order of l^2/R^2 the inequality $s_2 > 0$ is equivalent to $a_{33} > 0$, and conditions (2.5) (the first, third, and second, respectively) coincide with the corresponding stability conditions in the restricted formulation of the problem [1]

$$\begin{aligned} A_1 - A_3 \sin^2 \theta_0 - A_3 \cos^2 \theta_0 &> 0, & A_2 + \frac{k_3}{4\omega_0 \cos^2 \theta_0} &> A_3 \\ (A_1 - A_3 \sin^2 \theta_0 - A_3 \cos^2 \theta_0) \left(A_2 - A_1 + \frac{k_3}{\omega_0 \cos \theta_0} \right) &+ 3(A_1 - A_2)(A_3 - A_3) \sin^2 \theta_0 &> 0 \end{aligned}$$

Violation of conditions (2.5) ensures instability of steady motion (2.3), in which case the degree of instability is odd. This occurs when either $s_1 \neq 0, s_2 < 0, s_3 > 0$ or $s_1 \neq 0, s_2 > 0, s_3 < 0$.

3. In the case of a dynamically symmetric satellite (when $A_1 = A_2$ and $k_1 = k_2 = 0$) we have the additional cyclical coordinate φ . Elimination of the cyclical coordinates σ and φ by the Routh method brings us in this case to the altered potential energy [1]

$$W(R, \kappa, \theta, \psi) = 1/2 K^2 / S_1 - U$$

$$K = k - c\beta_3, S_1 = MR^2 \cos^2 \kappa + A_1(1 - \beta_3^2), \beta_3 = \cos \theta \sin \kappa - \sin \theta \cos \psi \cos \kappa$$

$$U = \mu M / R + \mu R^{-2} (A_1 - A_2)(1 - 1/2 \sin^2 \theta)$$

Here c is the constant of the cyclical integral corresponding to the cyclical coordinate φ ; the constant k_3 enters additively into c .

The steady motions of the satellite are defined by the equations

$$\begin{aligned} \frac{\partial W}{\partial R} &= -\frac{K^2}{S_1^2} MR \cos^2 \kappa + M \frac{\mu}{R^2} + 3 \frac{\mu}{R^2} (A_1 - A_2) \left(1 - \frac{3}{2} \sin^2 \theta\right) = 0 \\ \frac{\partial W}{\partial \kappa} &= -\frac{K}{S_1} c (\cos \theta \cos \kappa + \sin \theta \cos \psi \sin \kappa) + \frac{K^2}{S_1^2} \left[\frac{1}{2} MR^2 \sin 2\kappa + \right. \\ &\quad \left. + A_1 \beta_2 (\cos \theta \cos \kappa + \sin \theta \cos \psi \sin \kappa) \right] = 0 \tag{3.1} \\ \frac{\partial W}{\partial \theta} &= -\frac{K}{S_1} \left(\frac{K}{S_1} A_1 \beta_2 - c \right) (\sin \theta \sin \kappa + \cos \theta \cos \psi \cos \kappa) + \\ &\quad + \frac{3}{2} \frac{\mu}{R^2} (A_1 - A_2) \sin 2\theta = 0 \\ \frac{\partial W}{\partial \psi} &= \frac{K}{S_1} \left(\frac{K}{S_1} A_1 \beta_2 - c \right) \sin \theta \sin \psi \cos \kappa = 0 \end{aligned}$$

In addition to the solutions already considered [1] for $\kappa = 0$, Eqs. (3.1) have the solution

$$R = R_0, \quad \kappa = \kappa_0, \quad \theta = \theta_0, \quad \psi = \pi \tag{3.2}$$

if

$$\begin{aligned} -\omega_0 c \cos(\theta_0 + \kappa_0) + \frac{1}{2} A_1 \omega_0^2 \sin 2(\theta_0 + \kappa_0) + \frac{3}{2} \mu R_0^{-2} (A_1 - A_2) \sin 2\theta_0 &= 0 \\ \omega_0^2 \cos^2 \kappa_0 &= \frac{\mu}{R_0^2} \left[1 + \frac{3(A_1 - A_2)}{MR_0^2} \left(1 - \frac{3}{2} \sin^2 \theta_0\right) \right] \tag{3.3} \\ \sin 2\kappa_0 &= \frac{3(A_1 - A_2) \mu}{MR_0^2 \omega_0^2} \frac{\mu}{R_0^2} \sin 2\theta_0 \quad \left(\omega_0 = \omega_0' = \frac{K_0}{S_{10}} \right) \end{aligned}$$

In steady motion (3.2) (which constitutes a regular precession of the satellite in Koenig coordinates) the straight line OG forms a constant angle κ_0 with the plane $O\xi_0 \zeta_0$, and the axis of dynamic symmetry x_3 of the satellite lies in the plane $G\nu_2 \nu_3$, forming the angle θ_0 with the axis ν_3 . As in the previous case, the angle κ_0 is of the order of R/R^2 and is maximum for $\theta_0 = 1/2\pi$.

After computing the second partial derivatives of the function W for the values (3.2) we can readily deduce the fact that the second variation $\delta^2 W$ for real satellites is positive-definite provided the single inequality

$$\frac{\partial^2 W}{\partial \psi^2} = 3 \frac{\mu}{R_0^2} (A_1 - A_2) \frac{\sin^2 \theta_0 \cos \theta_0 \cos \kappa_0}{\cos(\theta_0 + \kappa_0)} > 0 \tag{3.4}$$

is fulfilled. This inequality is therefore the sufficient condition of stability of motion (3.2) with respect to $R, R', \kappa, \kappa', \theta, \theta', \psi, \psi', \sigma, \sigma'$ provided the constants k and c are unperturbed. For $|\theta_0| < 1/2\pi$ inequality (3.4) reduces to the condition $A_1 > A_2$, which coincides with the condition of stability in the restricted formulation of the problem. Violation of condition (3.4) with replacement of the inequality symbol by its opposite makes the unperturbed motion unstable.

4. The Routh theorem guarantees conditional stability. However, motions (2.3) and (3.2) are also unconditionally stable (when the constants k and c are perturbed), since the Liapunov addepdum [1,2] to the Routh theorem is valid for them by virtue of the theorem on implicit functions. In fact, the Jacobians of system (2.1), (2.2) and

system (3.1), given by

$$\frac{\cos^2 \alpha_0}{\cos^2 \theta_0} \alpha_1 \alpha_2, \quad \frac{\partial^2 W}{\partial \psi^2} \Delta \quad (\Delta \neq 0)$$

respectively, are different from zero by virtue of conditions (2.5) and (3.4), and the left sides of Eqs. (2.1), (2.2), and (3.1) together with their respective partial derivatives are continuous in the neighborhoods of values (2.3) and (3.2).

In general, the Jacobian of the equations of steady motions in the form (2.1), (2.2), or (3.1) coincides with the maximum minor in the criterion [2] for a conditional minimum of W or, respectively, in the Sylvester conditions for the positive definiteness of $\delta^2 W$. Hence, if we use either the criterion formulated in [2] (or some equivalent or coarser conditions) or the conditions of positive definiteness of $\delta^2 W$ in applying the Routh theorem (Theorems 2 and 4 of [1], pp. 16, 20), then the requirements of Liapunov's addendum [1,2] will be fulfilled.

The same can be said of Routh Theorems 1 and 1a of [1].

5. In each group of conditions (2.4) and (3.3) of existence of solutions (2.3) and (3.2) the first relation expresses the equality to zero of the sum of moments with respect to the y_1 axis of the gyroscopic, centrifugal, and gravitational forces applied to the satellite;

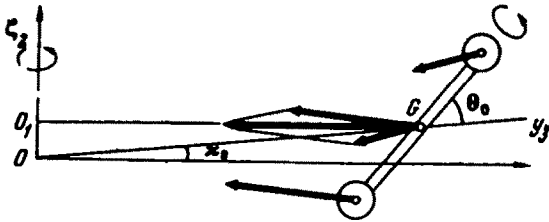


Fig. 1.

the second and third relations express the equality of the resultants of the centrifugal forces (with the opposite signs) and of the gravitational forces projected on the axes y_2 and y_3 (the second condition has been divided through by MR_0 and

the third by $MR_0 \sin \theta_0$). It should be noted that in the tilted position of the satellite which we are considering the sum of projections of the gravitational forces on the axis y_3 is equal to $\frac{3}{2} \mu R_0^{-4} (A_2 - A_3) \sin 2\theta_0$ and differs from zero. This results in the bias of the plane of motion of the center of mass. The example of a wobbling dumbbell-shaped satellite (see Fig. 1) shows this clearly.

The author is grateful to V. V. Rumiantsev for the formulation of the problem and his criticism.

BIBLIOGRAPHY

1. Rumiantsev, V. V., The stability of steady motions of satellites. Moscow, Vychisl. tsentr. Akad. Nauk, SSSR, 1967.
2. Shostak, R. Ia., On a criterion of conditional definiteness of a quadratic form of n variables under linear constraints and on the sufficient criterion for a conditional extremum of a function of n variables. Usp. Mat. Nauk Vol. 9, No. 2, 1954.
3. Liapunov, A. M., On steady twisting motions of a body in a fluid. Works, Vol. 1, Moscow, "Nauka", 1965.

Translated by A. Y.